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Generalized Darboux transformations for the KP equation with self-consistent sources

Ting Xiao and Yunbo Zeng

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China

E-mail: yzeng@math.tsinghua.edu.cn

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Abstract

The KP equation with self-consistent sources (KPESCS) is treated in the framework of the constrained KP equation. This offers a natural way to obtain the Lax representation for the KPESCS. Based on the conjugate Lax pairs, we construct the generalized binary Darboux transformation with arbitrary functions in time t for the KPESCS which, in contrast to the binary Darboux transformation of the KP equation, provides a non-auto-Bäcklund transformation between two KPESCSs with different degrees. The formula for N-times repeated generalized binary Darboux transformation is proposed and enables us to find the N-soliton solution and lump solution as well as some other solutions of the KPESCS.

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1. Introduction

Soliton equations with self-consistent sources (SESCSs) are important models in many fields of physics, such as hydrodynamics, solid state physics, plasma physics, etc [1–7]. Until now, much development has been made in the study of SESCS. For example, in the (1+1)-dimensional case, some SESCSs such as the KdV, modified KdV, nonlinear Schrödinger, AKNS and Kaup–Newell hierarchies with self-consistent sources were solved by the inverse scattering method [1–3, 6–10]. Recently, generalized binary Darboux transformations with arbitrary functions in time t for some (1+1)-dimensional SESCSs, which offer a non-auto-Bäcklund transformation between two SESCSs with different degrees of sources, have been constructed and can be used to obtain the N-soliton solution [11–13]. But in the (2+1)-dimensional case, fewer results for the SESCSs have been obtained. The KP equation with self-consistent sources (KPESCS) arose in some physical models describing the interaction of long and short waves and the soliton solution of it was first found by Mel'nikov [14, 15].

However, since the explicit time part of the Lax representation of the KPESCS was not found, the method for solving the KPESCS by inverse scattering transformation in [14, 15] was quite complicated. Recently, an N-soliton solution of the KPESCS was obtained by the Hirota method in [16]. In this paper, we treat the constrained KP hierarchy as the stationary equations of the KP hierarchy with self-consistent sources. This gives a natural way to find the Lax pair for the KPESCS. Using the conjugate Lax pairs, we construct the generalized binary Darboux transformation with arbitrary functions in time t for the KPESCS. In contrast to the binary Darboux transformation for the KP equation which offers Bäcklund transformation, this transformation provides a non-auto-Bäcklund transformation between two KPESCSs with different degrees of sources. Some interesting solutions of KPESCS such as the soliton solution, lump solution and mixture solution of exponential and rational solutions are obtained by this generalized Darboux transformation.

The paper will be organized as follows. We recall some facts about the Darboux transformation for the KP equation in the next section. In section 3, through the pseudo-differential operator (PDO) formalism we reveal the relation between the KP hierarchy with self-consistent sources and the constrained KP hierarchy. Then the conjugate Lax pairs of the KP hierarchy with self-consistent sources can be obtained naturally. Using the conjugate Lax pairs, we can construct the generalized Darboux transformations with arbitrary functions in time for KPESCS and find some interesting solutions of the KPESCS. In section 4, the N-times repeated generalized Darboux transformation will be constructed by which the N-soliton solution and some other solutions can be obtained.

2. The Darboux transformation for the KP equation

We first give a simple description of the KP hierarchy within the framework of the Sato theory (see [20, 21]). Let us consider the following pseudo-differential operator (PDO)

$$L = \partial + u_0\partial^{-1} + u_1\partial^{-2} + \cdots \tag{2.1}$$

where ∂ denotes $\frac{\partial}{\partial x}$, and u_j, $j = 0, 1, \ldots$ are functions. Denote $B_m = (L^m)_+$ for $\forall m \in N$ where $(L^m)_+$ represents the projection of L^m to its pure differential part. Then the KP hierarchy has the following Lax representation (zero-curvature representation):

$$(B_n)_{t_k} - (B_k)_{t_n} + [B_n, B_k] = 0 \qquad n, k \geqslant 2. \tag{2.2}$$

Equation (2.2) has a pair of conjugate Lax pairs as follows:

$$\psi^-_{t_k} = B_k\psi^- \tag{2.3a}$$

$$\psi^-_{t_n} = B_n\psi^- \tag{2.3b}$$

and

$$\psi^+_{t_k} = -(B_k)^*\psi^+ \tag{2.4a}$$

$$\psi^+_{t_n} = -(B_n)^*\psi^+. \tag{2.4b}$$

When $k = 2, n = 3$ and under the following transformation:

$$u = 2u_0 \qquad t = -\frac{1}{4}t_3 \qquad y = \alpha t_2 \tag{2.5}$$

we get the simplest and the most important equation in the hierarchy (2.2), the KP equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\alpha^2 u_{yy} = 0 \tag{2.6}$$

which appears in physical applications in two different forms with $\alpha = 1$ and $\alpha = i$, usually referred to as the KPI and KPII equation [17]. Under transformation (2.5), we obtain the conjugate Lax pairs of (2.6) respectively from (2.4) and (2.3) as follows:

$$\alpha\psi^+_y = -\psi^+_{xx} - u\psi^+ \tag{2.7a}$$

$$\psi^+_t = A^+(u)\psi^+ \qquad A^+(u) = -4\partial^3 - 6u\partial - 3(u_x - \alpha\partial^{-1}u_y) \tag{2.7b}$$

and

$$\alpha\psi^-_y = \psi^-_{xx} + u\psi^- \tag{2.8a}$$

$$\psi^-_t = A^-(u)\psi^- \qquad A^-(u) = -4\partial^3 - 6u\partial - 3(u_x + \alpha\partial^{-1}u_y). \tag{2.8b}$$

From (2.7) and (2.8), we can construct three types of Darboux transformations for the KP equation (2.6).

(1) The forward Darboux transformation for the KP equation.
Assume that u is a solution of the KP equation (2.6) and denote a fixed solution of (2.7) by $\psi^+_1 = \psi^+_1(x, y, t)$. The forward Darboux transformation for (2.7) is given by [19]

$$\psi^+[+1] = \psi^+_x - \frac{\psi^+_{1,x}}{\psi^+_1}\psi^+ \tag{2.9a}$$

$$u[+1] = u + 2\partial^2\ln\psi^+_1. \tag{2.9b}$$

So $u[+1]$ is a new solution of the KP equation (2.6). Substituting (2.9) into (2.7b), we have

$$\begin{aligned} A^+(u[+1])\psi^+[+1] &= \left[\psi^+_x - \frac{\psi^+_{1,x}}{\psi^+_1}\psi^+\right]_t \\ &= \psi^+_{xt} - \left(\frac{\psi^+_{1,x}}{\psi^+_1}\psi^+\right)_t \\ &= (A^+(u)\psi^+)_x - \left(\frac{A^+(u)\psi^+_1}{\psi^+_1}\right)_x\psi^+ - \left(\frac{\psi^+_{1,x}}{\psi^+_1}\right)A^+(u)\psi^+. \end{aligned} \tag{2.10}$$

(2) The backward Darboux transformation for the KP equation.
Assume that u is a solution of the KP equation (2.6) and denote a fixed solution of (2.8) by $\psi^-_2 = \psi^-_2(x, y, t)$. The backward Darboux transformation for the system (2.7) is defined by

$$\psi^+[-1] = \frac{a_1 + \int \psi^+\psi^-_2\,\mathrm{d}x}{\psi^-_2} \tag{2.11a}$$

$$u[-1] = u + 2\partial^2\ln\psi^-_2 \tag{2.11b}$$

where a_1 is an arbitrary constant. We point out that throughout the paper, the integral operation $\int f_1 f_2\,\mathrm{d}x$ such as $\int \psi^+\psi^-_2\,\mathrm{d}x$ here means $\int_{-\infty}^x f_1 f_2\,\mathrm{d}x$ or $-\int_x^\infty f_1 f_2\,\mathrm{d}x$ and contains no arbitrary function of y and t, only numerical constant if we impose some suitable boundary condition on the integrand functions f_1 and f_2 at $x = -\infty$ or $x = \infty$. For arbitrariness of the constants in the Darboux transformations such as a_1 here, in our computation later, the integral constants are taken to be zero.

Substituting (2.11) into (2.7b), we get the following equality:

$$A^+(u[-1])\psi^+[-1] = \left(\frac{a_1 + \int \psi^+\psi_2^- \,\mathrm{d}x}{\psi_2^-}\right)_t$$
$$= -\frac{A^-(u)\psi_2^-}{(\psi_2^-)^2}\left(a_1 + \int \psi^+\psi_2^- \,\mathrm{d}x\right)$$
$$+ \frac{\int[\psi^+(A^-(u)\psi_2^-) + \psi_2^-(A^+(u)\psi^+)]\,\mathrm{d}x}{\psi_2^-}. \qquad (2.12)$$

(3) The binary Darboux transformation for the KP equation.
When the backward Darboux transformation and forward Darboux transformation are applied consecutively to the system (2.7), we can get the binary DT as follows:

$$\psi^+[-1,+1] = \psi^+[-1]_x - \frac{\psi_1^+[-1]_x}{\psi_1^+[-1]}\psi^+[-1] = \psi^+ - \frac{\psi_1^+\left(a_1 + \int \psi^+\psi_2^- \,\mathrm{d}x\right)}{a_2 + \int \psi_2^-\psi_1^+ \,\mathrm{d}x} \qquad (2.13a)$$

$$u[-1,+1] = u[-1] + 2\partial^2 \ln \psi_1^+[-1] = u + 2\partial^2 \ln\left(a_2 + \int \psi_2^-\psi_1^+ \,\mathrm{d}x\right) \qquad (2.13b)$$

where $\psi_1^+[-1] = \frac{a_2 + \int \psi_1^+\psi_2^- \mathrm{d}x}{\psi_2^-}$, a_j, $j = 1, 2$ are arbitrary constants.

Remark. We can define the forward DT, backward DT and binary DT for system (2.8) analogously. We list the results here:

Forward DT of (2.8),

$$\psi^- \to \psi^-[-1] = \psi_x^- - \frac{\psi_{2,x}^-}{\psi_2^-}\psi^- \qquad (2.14a)$$

$$u \to u[-1] = u + 2\partial^2 \ln \psi_2^-. \qquad (2.14b)$$

Backward DT of (2.8),

$$\psi^- \to \psi^-[+1] = \frac{a_3 + \int \psi^-\psi_1^+ \,\mathrm{d}x}{\psi_1^+} \qquad (2.15a)$$

$$u \to u[+1] = u + 2\partial^2 \ln \psi_1^+. \qquad (2.15b)$$

Binary DT of (2.8),

$$\psi^- \to \psi^-[+1,-1] = \psi^- - \frac{\psi_1^+\left(a_3 + \int \psi^-\psi_1^+ \,\mathrm{d}x\right)}{a_2 + \int \psi_2^-\psi_1^+ \,\mathrm{d}x} \qquad (2.16a)$$

$$u \to u[+1,-1] = u + 2\partial^2 \ln\left(a_2 + \int \psi_1^+\psi_2^- \,\mathrm{d}x\right) \qquad (2.16b)$$

where $\psi_2^-[+1] = \frac{a_2 + \int \psi_1^+\psi_2^- \,\mathrm{d}x}{\psi_1^+}$, a_j, $j = 2, 3$ are arbitrary constants.

3. The Darboux transformation for the KP equation with self-consistent sources

We first recall the constrained KP hierarchy (for more details refer to [22–26] and the references therein). For the pseudo-differential operator L given by (2.1), if we impose the condition that

$(q_j)_{t_k} = B_k q_j$, $(r_j)_{t_k} = -B_k^* q_j$, $j = 1, \ldots, m$, $k \geqslant 2$, and make the constraint

$$(L^n)_- = \sum_{j=1}^{m} q_j \partial^{-1} r_j \qquad \text{or} \qquad L^n = B_n + \sum_{j=1}^{m} q_j \partial^{-1} r_j$$

the n-constrained KP hierarchy is defined as follows:

$$(L^n)_{t_k} = [B_k, L^n] \tag{3.1a}$$

$$(q_j)_{t_k} = B_k q_j \tag{3.1b}$$

$$(r_j)_{t_k} = -B_k^* r_j \qquad j = 1, \ldots, m. \tag{3.1c}$$

If the term $(B_k)_{t_n}$ is added to the right-hand side of equation (3.1*a*), we get the KP hierarchy with self-consistent sources as

$$(B_k)_{t_n} - (L^n)_{t_k} + [B_k, L^n] = 0 \tag{3.2a}$$

$$(q_j)_{t_k} = B_k q_j \tag{3.2b}$$

$$(r_j)_{t_k} = -B_k^* r_j \qquad j = 1, \ldots, m. \tag{3.2c}$$

If the variable 't_n' is viewed as the evolution variable, the n-constrained KP hierarchy (3.1) may be considered as the stationary case (i.e. $(B_k)_{t_n} = 0$) of the KP hierarchy with self-consistent sources (3.2). Under conditions (3.2*b*) and (3.2*c*), we naturally get the conjugate Lax pairs for (3.2*a*) as

$$\psi^-_{t_k} = B_k \psi^- \tag{3.3a}$$

$$\psi^-_{t_n} = L^n \psi^- = B_n \psi^- + \sum_{j=1}^{m} q_j \int r_j \psi^- \, \mathrm{d}x \tag{3.3b}$$

and

$$\psi^+_{t_k} = -B_k^* \psi^+ \tag{3.4a}$$

$$\psi^+_{t_n} = -(L^n)^* \psi^+ = -B_n^* \psi^+ + \sum_{j=1}^{m} r_j \int q_j \psi^+ \, \mathrm{d}x. \tag{3.4b}$$

When $k = 2$, $n = 3$, under transformation (2.5) and $\Psi_j = -r_j$, $\Phi_j = q_j$, we will get the KPESCS [14] from (3.2) as follows:

$$\left[u_t + 6uu_x + u_{xxx} + 8 \left(\sum_{j=1}^{m} \Phi_j \Psi_j \right)_x \right]_x + 3\alpha^2 u_{yy} = 0 \tag{3.5a}$$

$$\alpha \Phi_{j,y} = \Phi_{j,xx} + u \Phi_j \tag{3.5b}$$

$$\alpha \Psi_{j,y} = -\Psi_{j,xx} - u \Psi_j \qquad j = 1, \ldots, m \tag{3.5c}$$

and its conjugate Lax pairs from (3.4) and (3.3) respectively as follows:

$$\alpha \psi^+_y = -\psi^+_{xx} - u \psi^+ \tag{3.6a}$$

$$\psi^+_t = A^+(u)\psi^+ + T^+_m(\Psi, \Phi)\psi^+ \qquad T^+_m(\Psi, \Phi)\psi^+ = 4 \sum_{j=1}^{m} \Psi_j \int \Phi_j \psi^+ \, \mathrm{d}x \tag{3.6b}$$

and

$$\alpha\psi_y^- = \psi_{xx}^- + u\psi^- \tag{3.7a}$$

$$\psi_t^- = A^-(u)\psi^- + T_m^-(\Psi,\Phi)\psi^- \qquad T_m^-(\Psi,\Phi)\psi^- = 4\sum_{j=1}^{m}\Phi_j\int\Psi_j\psi^-\,\mathrm{d}x. \tag{3.7b}$$

From the conjugate Lax pairs for KPESCS (3.5), we can also construct the forward DT, backward DT and binary DT for it.

Theorem 3.1. *Assume $u, \Phi_1, \ldots, \Phi_m, \Psi_1, \ldots, \Psi_m$ be the solution of the KPESCS (3.5) and ψ_1^+ satisfies (3.6), then the forward Darboux transformation for (3.6) can be defined by*

$$\psi^+[+1] = \psi_x^+ - \frac{\psi_{1,x}^+}{\psi_1^+}\psi^+ \tag{3.8a}$$

$$u[+1] = u + 2\partial^2 \ln\psi_1^+ \tag{3.8b}$$

$$\Phi_j[+1] = -\frac{\int\psi_1^+\Phi_j\,\mathrm{d}x}{\psi_1^+} \tag{3.8c}$$

$$\Psi_j[+1] = \Psi_{j,x} - \frac{\psi_{1,x}^+}{\psi_1^+}\Psi_j \qquad j = 1,\ldots,m \tag{3.8d}$$

namely, $u[+1], \psi^+[+1], \Psi_j[+1], \Phi_j[+1], j = 1,\ldots,m$, satisfy (3.5) and (3.6).

Proof. Based on the results in the previous section, it is obvious that (3.5*b*), (3.5*c*) and (3.6*a*) hold under transformation (3.8). So we only need to prove (3.6*b*), i.e., the following equality,

$$\begin{aligned}
\psi^+[+1]_t &= \left[\psi_x^+ - \frac{\psi_{1,x}^+}{\psi_1^+}\psi^+\right]_t \\
&= \left(A^+(u)\psi^+ + T_m^+(\Psi,\Phi)\psi^+\right)_x - \left(\frac{A^+(u)\psi_1^+ + T_m^+(\Psi,\Phi)\psi_1^+}{\psi_1^+}\right)_x\psi^+ \\
&\quad - \frac{\psi_{1,x}^+}{\psi_1^+}\left(A^+(u)\psi^+ + T_m^+(\Psi,\Phi)\psi^+\right) \\
&= (A^+(u)\psi^+)_x - \left(\frac{A^+(u)\psi_1^+}{\psi_1^+}\right)_x\psi^+ - \frac{\psi_{1,x}^+}{\psi_1^+}A^+(u)\psi^+ \\
&\quad + \left(T_m^+(\Psi,\Phi)\psi^+\right)_x - \left(\frac{T_m^+(\Psi,\Phi)\psi_1^+}{\psi_1^+}\right)_x\psi^+ - \frac{\psi_{1,x}^+}{\psi_1^+}T_m^+(\Psi,\Phi)\psi^+ \\
&= A^+(u[+1])\psi^+[+1] + T_m^+(\Psi[+1],\Phi[+1])\psi^+[+1].
\end{aligned} \tag{3.9}$$

By simple computation, we can prove that equality (2.10) still holds now, so we only need to check the terms containing Φ_j, Ψ_j in equality (3.9), i.e., to check the following equality:

$$\left(T_m^+(\Psi,\Phi)\psi^+\right)_x - \left(\frac{T_m^+(\Psi,\Phi)\psi_1^+}{\psi_1^+}\right)_x\psi^+ - \frac{\psi_{1,x}^+}{\psi_1^+}T_m^+(\Psi,\Phi)\psi^+ = T_m^+(\Psi[+1],\Phi[+1])\psi^+[+1]. \tag{3.10}$$

In fact, we have

$$
\begin{aligned}
\text{the lhs of (3.10)} &= 4\left[\sum_{j=1}^{m}\Psi_j\int\Phi_j\psi^+\,\mathrm{d}x\right]_x - 4\left[\frac{\sum_{j=1}^{m}\Psi_j\int\Phi_j\psi_1^+\,\mathrm{d}x}{\psi_1^+}\right]_x\psi^+\\
&\quad - 4\frac{\psi_{1,x}^+}{\psi_1^+}\sum_{j=1}^{m}\Psi_j\int\Phi_j\psi^+\,\mathrm{d}x\\
&= 4\sum_{j=1}^{m}\left[\Psi_{j,x}\int\Phi_j\psi^+\,\mathrm{d}x + \Psi_j\Phi_j\psi^+\right] - 4\frac{\psi_{1,x}^+}{\psi_1^+}\sum_{j=1}^{m}\Psi_j\int\Phi_j\psi^+\,\mathrm{d}x\\
&\quad - 4\frac{\sum_{j=1}^{m}\left[\left(\Psi_{j,x}\int\Phi_j\psi_1^+\,\mathrm{d}x + \Psi_j\Phi_j\psi_1^+\right)\psi_1^+ - \left(\Psi_j\int\Phi_j\psi_1^+\,\mathrm{d}x\right)\psi_{1,x}^+\right]}{\left(\psi_1^+\right)^2}\psi^+\\
&= 4\sum_{j=1}^{m}\left[\Psi_{j,x}\int\Phi_j\psi^+\,\mathrm{d}x - \frac{\psi^+\Psi_{j,x}}{\psi_1^+}\int\Phi_j\psi_1^+\,\mathrm{d}x\right.\\
&\quad\left. + \Psi_j\psi^+\psi_{1,x}^+\frac{\int\Phi_j\psi_1^+\,\mathrm{d}x}{\left(\psi_1^+\right)^2} - \frac{\psi_{1,x}^+}{\psi_1^+}\Psi_j\int\Phi_j\psi^+\,\mathrm{d}x\right]\\
&= 4\sum_{j=1}^{m}\left[\left(\Psi_{j,x} - \frac{\psi_{1,x}^+}{\psi_1^+}\Psi_j\right)\int\Phi_j\psi^+\,\mathrm{d}x - \frac{\psi^+\int\Phi_j\psi_1^+\,\mathrm{d}x}{\psi_1^+}\left(\Psi_{j,x} - \frac{\psi_{1,x}^+}{\psi_1^+}\Psi_j\right)\right]\\
&= 4\sum_{j=1}^{m}\Psi_j[+1]\left(\int\Phi_j\psi^+\,\mathrm{d}x - \frac{\psi^+}{\psi_1^+}\int\Phi_j\psi_1^+\,\mathrm{d}x\right)
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
\text{the rhs of (3.10)} &= 4\sum_{j=1}^{m}\Psi_j[+1]\int\Phi_j[+1]\psi^+[+1]\,\mathrm{d}x\\
&= 4\sum_{j=1}^{m}\Psi_j[+1]\int\left[-\frac{\int\psi_1^+\Phi_j\,\mathrm{d}x}{\psi_1^+}\left(\psi_x^+ - \frac{\psi_{1,x}^+}{\psi_1^+}\psi^+\right)\right]\mathrm{d}x\\
&= -4\sum_{j=1}^{m}\Psi_j[+1]\int\left[\left(\int\psi_1^+\Phi_j\,\mathrm{d}x\right)\mathrm{d}\left(\frac{\psi^+}{\psi_1^+}\right)\right]\mathrm{d}x\\
&= 4\sum_{j=1}^{m}\Psi_j[+1]\left[\int\Phi_j\psi^+\,\mathrm{d}x - \frac{\psi^+}{\psi_1^+}\int\Phi_j\psi_1^+\,\mathrm{d}x\right]\\
&= \text{the lhs of (3.10).}
\end{aligned}
\tag{3.12}
$$

This completes the proof. □

Theorem 3.2. *Assume $u, \Phi_1, \ldots, \Phi_m, \Psi_1, \ldots, \Psi_m$ be a solution of the KPESCS (3.5) and ψ_2^- satisfies (3.7), then the backward DT for (3.6) is defined by*

$$\psi^+[-1] = \frac{a_4 + \int\psi_2^-\psi^+\,\mathrm{d}x}{\psi_2^-} \tag{3.13a}$$

$$u[-1] = u + 2\partial^2\ln\psi_2^- \tag{3.13b}$$

$$\Phi_j[-1] = \Phi_{j,x} - \frac{\psi_{2,x}^-}{\psi_2^-}\Phi_j \tag{3.13c}$$

$$\Psi_j[-1] = -\frac{\int \psi_2^- \Psi_j \,\mathrm{d}x}{\psi_2^-} \qquad j = 1, \ldots, m \tag{3.13d}$$

where a_4 is an arbitrary constant, namely $u[-1]$, $\psi^+[-1]$, $\Psi_j[-1]$, $\Phi_j[-1]$, $j = 1, \ldots, m$, satisfy (3.5) and (3.6).

Proof. It is obvious that (3.5*b*), (3.5*c*) and (3.6*a*) hold under transformation (3.13). So we only need to prove (3.6*b*), i.e., to prove the following equality:

$$\begin{aligned}
\psi^+[-1]_t &= \left[\frac{a_4 + \int \psi_2^- \psi^+ \,\mathrm{d}x}{\psi_2^-}\right]_t \\
&= -\frac{A^-(u)\psi_2^-}{(\psi_2^-)^2}\left(a_4 + \int \psi_2^-\psi^+ \,\mathrm{d}x\right) + \frac{\int [(A^-(u)\psi_2^-)\psi^+ + (A^+(u)\psi^+)\psi_2^-]\,\mathrm{d}x}{\psi_2^-} \\
&\quad - \frac{T_m^-(\Phi,\Psi)\psi_2^-}{(\psi_2^-)^2}\left(a_4 + \int \psi_2^+\psi^+ \,\mathrm{d}x\right) \\
&\quad + \frac{\int \left[(T_m^-(\Phi,\Psi)\psi_2^-)\psi^+ + \left(T_m^+(\Psi,\Phi)\psi^+\right)\psi_2^-\right]\mathrm{d}x}{\psi_2^-} \\
&= A^+(u[-1])\psi^+[-1] + T_m^+(\Psi[-1],\Phi[-1])\psi^+[-1].
\end{aligned} \tag{3.14}$$

Similarly, using equality (2.12), we only need to check the terms containing Ψ_j, Φ_j, i.e.,

$$\begin{aligned}
&-\frac{T_m^-(\Psi,\Phi)\psi_2^-}{(\psi_2^-)^2}\left(a_4 + \int \psi_2^-\psi^+ \,\mathrm{d}x\right) + \frac{\int \left[(T_m^-(\Psi,\Phi)\psi_2^-)\psi^+ + \left(T_m^+(\Psi,\Phi)\psi^+\right)\psi_2^-\right]\mathrm{d}x}{\psi_2^-} \\
&\quad = T_m^+(\Psi[-1],\Phi[-1])\psi^+[-1]
\end{aligned} \tag{3.15}$$

i.e.,

$$\begin{aligned}
&4\sum_{j=1}^m \left\{-\frac{\Phi_j \int \psi_2^-\Psi_j \,\mathrm{d}x}{(\psi_2^-)^2}\left(a_4 + \int \psi_2^-\psi^+ \,\mathrm{d}x\right) + \frac{\int \left[\psi^+\Phi_j \int(\psi_2^-\Psi_j) + \psi_2^-\Psi_j \int(\psi^+\Phi_j)\right]\mathrm{d}x}{\psi_2^-}\right\} \\
&\quad = 4\sum_{j=1}^m \Psi_j[-1]\int \Phi_j[-1]\psi^+[-1]\,\mathrm{d}x.
\end{aligned} \tag{3.16}$$

In fact, we have

$$\begin{aligned}
\text{the lhs of (3.16)} &= 4\sum_{j=1}^m \left[-\frac{\Phi_j \int \psi_2^-\Psi_j \,\mathrm{d}x}{(\psi_2^-)^2}\left(a_4 + \int \psi_2^-\psi^+ \,\mathrm{d}x\right) + \frac{\int \psi^+\Phi_j \,\mathrm{d}x \int \psi_2^-\Psi_j \,\mathrm{d}x}{\psi_2^-}\right] \\
&= 4\sum_{j=1}^m \frac{\int \Psi_j\psi_2^- \,\mathrm{d}x}{\psi_2^-}\left[-\frac{\Phi_j}{\psi_2^-}\left(a_4 + \int \psi_2^-\psi^+ \,\mathrm{d}x\right) + \int \psi^+\Phi_j \,\mathrm{d}x\right] \\
&= -4\sum_{j=1}^m \frac{\int \Psi_j\psi_2^- \,\mathrm{d}x}{\psi_2^-}\left\{\int \left[\left(a_4 + \int \psi_2^-\psi^+ \,\mathrm{d}x\right) d\left(\frac{\Phi_j}{\psi_2^-}\right)\right]\mathrm{d}x\right\}
\end{aligned}$$

$$= -4\sum_{j=1}^{m}\frac{\int \Psi_j\psi_2^-\,\mathrm{d}x}{\psi_2^-}\int\left[\frac{\left(a_4+\int\psi_2^-\psi^+\,\mathrm{d}x\right)}{\psi_2^-}\frac{(\Phi_{j,x}\psi_2^- - \Phi_j\psi_{2,x}^-)}{\psi_2^-}\right]\mathrm{d}x$$
$$= 4\sum_{j=1}^{m}\Psi_j[-1]\int\psi^+[-1]\Phi_j[-1]\,\mathrm{d}x$$
$$= \text{the rhs of (3.16).} \tag{3.17}$$

This completes the proof. □

From theorem 3.1 and theorem 3.2, we can obtain the binary Darboux transformation for the system (3.6) by choosing $a_4 = 0$, $\psi_1^+[-1] = \frac{C+\int\psi_1^+\psi_2^-\,\mathrm{d}x}{\psi_2^-}$ (C is a constant) as follows:

$$\psi^+[-1,+1] = \psi^+ - \frac{\psi_1^+\int\psi_2^-\psi^+\,\mathrm{d}x}{C+\int\psi_1^+\psi_2^-\,\mathrm{d}x} \tag{3.18a}$$

$$u[-1,+1] = u + 2\partial^2\ln\left(C+\int\psi_2^-\psi_1^+\,\mathrm{d}x\right) \tag{3.18b}$$

$$\Phi_j[-1,+1] = \Phi_j - \frac{\psi_2^-\int\psi_1^+\Phi_j\,\mathrm{d}x}{C+\int\psi_2^-\psi_1^+\,\mathrm{d}x} \tag{3.18c}$$

$$\Psi_j[-1,+1] = \Psi_j - \frac{\psi_1^+\int\psi_2^-\Psi_j\,\mathrm{d}x}{C+\int\psi_2^-\psi_1^+\,\mathrm{d}x} \qquad j = 1,\ldots,m. \tag{3.18d}$$

Substituting (3.18) into equation (3.6b) gives the equality

$$\psi^+[-1,+1]_t = A^+(u[-1,+1])\psi^+[-1,+1] + T_m^+(\Psi[-1,+1],\Phi[-1,+1])\psi^+[-1,+1]. \tag{3.19}$$

Both sides of equality (3.19) are polynomials w.r.t. the term $\left(C+\int\psi_2^-\psi_1^+\,\mathrm{d}x\right)^{-1}$. For example, the left-hand side of (3.19) is a polynomial of order 2 w.r.t. $\left(C+\int\psi_2^-\psi_1^+\,\mathrm{d}x\right)^{-1}$ as follows:

$$\psi^+[-1,+1]_t = \left[\psi^+ - \frac{\psi_1^+\int\psi_2^-\psi^+\,\mathrm{d}x}{C+\int\psi_1^+\psi_2^-\,\mathrm{d}x}\right]_t$$
$$= \psi_t^+ - \frac{\psi_{1,t}^+\left(\int\psi_2^-\psi^+\,\mathrm{d}x\right)}{C+\int\psi_1^+\psi_2^-\,\mathrm{d}x}$$
$$- \frac{\psi_1^+\left(\int\psi_2^-\psi^+\,\mathrm{d}x\right)_t}{C+\int\psi_1^+\psi_2^-\,\mathrm{d}x} + \frac{\psi_1^+\left(\int\psi_2^-\psi^+\,\mathrm{d}x\right)\left(\int\psi_2^-\psi^+\,\mathrm{d}x\right)_t}{\left(C+\int\psi_1^+\psi_2^-\,\mathrm{d}x\right)^2}$$
$$\equiv \sum_{j=0}^{2}L_j\left[C+\int\psi_2^-\psi_1^+\,\mathrm{d}x\right]^{-j}. \tag{3.20}$$

By a tedious computation, the right-hand side of (3.19) is displayed to be a polynomial of order less than 4 w.r.t. $\left(C+\int\psi_2^-\psi_1^+\,\mathrm{d}x\right)^{-1}$. We denote it by $\sum_{j=0}^{4}R_j\left[C+\int\psi_2^-\psi_1^+\,\mathrm{d}x\right]^{-j}$. Since (3.19) holds for the arbitrary constant C and L_j, R_j never contain C, we have

$$L_j = R_j \quad j = 0,1,2 \qquad \text{and} \qquad R_j = 0 \quad j = 3,4.$$

If we replace C in (3.18) by $C(t)$, an arbitrary function in t, and substitute equations (3.18) into both sides of the equations of (3.6) again, we will find that (3.6a) is also covariant w.r.t. (3.18). The right-hand side of (3.19) turns to be $\sum_{j=0}^{2} R_j\big[C(t) + \int \psi_2^- \psi_1^+ \,\mathrm{d}x\big]^{-j}$, but the left-hand side of (3.19) does not equal $\sum_{j=0}^{2} L_j\big[C(t) + \int \psi_2^- \psi_1^+ \,\mathrm{d}x\big]^{-j}$ any more. In other words, (3.6*b*) is not covariant w.r.t. (3.18) any longer when C is replaced by $C(t)$. In fact we have the following theorem.

Theorem 3.3. *Given $u, \Psi_1, \ldots, \Psi_m, \Phi_1, \ldots, \Phi_m$ is a solution of the KPESCS (3.5) and let ψ_1^+ and ψ_2^- be solutions of the system (3.6) and (3.7) respectively, then the transformation with $C(t)$, an arbitrary function in t defined by*

$$\psi^+[-1,+1] = \psi^+ - \frac{\psi_1^+ \int \psi_2^- \psi^+ \,\mathrm{d}x}{C(t) + \int \psi_1^+ \psi_2^- \,\mathrm{d}x} \tag{3.21a}$$

$$u[-1,+1] = u + 2\partial^2 \ln\left(C(t) + \int \psi_2^- \psi_1^+ \,\mathrm{d}x\right) \tag{3.21b}$$

$$\Phi_j[-1,+1] = \Phi_j - \frac{\psi_2^- \int \psi_1^+ \Phi_j \,\mathrm{d}x}{C(t) + \int \psi_2^- \psi_1^+ \,\mathrm{d}x} \tag{3.21c}$$

$$\Psi_j[-1,+1] = \Psi_j - \frac{\psi_1^+ \int \psi_2^- \Psi_j \,\mathrm{d}x}{C(t) + \int \psi_2^- \psi_1^+ \,\mathrm{d}x} \qquad j = 1, \ldots, m \tag{3.21d}$$

and

$$\Psi_{m+1}[-1,+1] = \frac{1}{2}\frac{\sqrt{\dot{C}(t)}\,\psi_1^+}{C(t) + \int \psi_2^- \psi_1^+ \,\mathrm{d}x} \qquad \Phi_{m+1}[-1,+1] = \frac{1}{2}\frac{\sqrt{\dot{C}(t)}\,\psi_2^-}{C(t) + \int \psi_2^- \psi_1^+ \,\mathrm{d}x} \tag{3.21e}$$

transforms (3.5b), (3.5c) and (3.6) respectively into

$$\alpha \Phi_j[-1,+1]_y = \Phi_j[-1,+1]_{xx} + u[-1,+1]\Phi_j[-1,+1] \tag{3.22a}$$

$$\alpha \Psi_j[-1,+1]_y = -\Psi_j[-1,+1]_{xx} - u[-1,+1]\Psi_j[-1,+1] \qquad j = 1, \ldots, m+1 \tag{3.22b}$$

$$\alpha \psi^+[-1,+1]_y = -\psi^+[-1,+1]_{xx} - u[-1,+1]\psi^+[-1,+1] \tag{3.22c}$$

$$\psi^+[-1,+1]_t = A^+(u[-1,+1])\psi^+[-1,+1] + T_{m+1}^+(\Psi[-1,+1], \Phi[-1,+1])\psi^+[-1,+1]. \tag{3.22d}$$

So $u[-1,+1], \Psi_1[-1,+1], \ldots, \Psi_{m+1}[-1,+1], \Phi_1[-1,+1], \ldots, \Phi_{m+1}[-1,+1]$ is a solution of the KPESCS (3.5) with degree $m+1$.

Proof. Equations (3.22*a*), (3.22*b*) and (3.22*c*) hold obviously. We only need to prove (3.22*d*). Substituting (3.21*a*) into the left-hand side of (3.22*d*), we have

$$\begin{aligned}
\psi^+[-1,+1]_t &= \left[\psi^+ - \frac{\psi_1^+ \int \psi_2^- \psi^+ \,\mathrm{d}x}{C(t) + \int \psi_1^+ \psi_2^- \,\mathrm{d}x}\right]_t \\
&= \sum_{j=0}^{2} L_j \left[C(t) + \int \psi_2^- \psi_1^+ \,\mathrm{d}x\right]^{-j} + \frac{\dot{C}(t)\psi_1^+ \int \psi^+ \psi_2^- \,\mathrm{d}x}{\left(C(t) + \int \psi_1^+ \psi_2^- \,\mathrm{d}x\right)^2} \\
&= \sum_{j=0}^{2} R_j \left[C(t) + \int \psi_2^- \psi_1^+ \,\mathrm{d}x\right]^{-j} + \frac{\dot{C}(t)\psi_1^+ \int \psi^+ \psi_2^- \,\mathrm{d}x}{\left(C(t) + \int \psi_1^+ \psi_2^- \,\mathrm{d}x\right)^2} \\
&= A^+(u[-1,+1])\psi^+[-1,+1] + T_m^+(\Psi[-1,+1], \Phi[-1,+1])\psi^+[-1,+1] \\
&\quad + \frac{\dot{C}(t)\psi_1^+ \int \psi^+ \psi_2^- \,\mathrm{d}x}{\left(C(t) + \int \psi_1^+ \psi_2^- \,\mathrm{d}x\right)^2}.
\end{aligned} \tag{3.23}$$

So we only need to prove

$$4\Psi_{m+1}[-1,+1]\int \Phi_{m+1}[-1,+1]\psi^+[-1,+1]\,\mathrm{d}x = \frac{\dot{C}(t)\psi_1^+\int\psi^+\psi_2^-\,\mathrm{d}x}{\left(C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x\right)^2}. \tag{3.24}$$

In fact, we have

$$\begin{aligned}
\text{the lhs of (3.24)} &= 4\times\frac{1}{2}\frac{\sqrt{\dot{C}(t)}\psi_1^+}{C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x}\\
&\quad\times\int\left[\left(\psi^+ - \frac{\psi_1^+\int\psi^+\psi_2^-\,\mathrm{d}x}{C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x}\right)\times\frac{1}{2}\frac{\sqrt{\dot{C}(t)}\psi_2^-}{C(t)+\int\psi_2^-\psi_1^+\,\mathrm{d}x}\right]\mathrm{d}x\\
&= \frac{\dot{C}(t)\psi_1^+}{C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x}\int\left[\left(\psi^+ - \frac{\psi_1^+\int\psi^+\psi_2^-\,\mathrm{d}x}{C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x}\right)\frac{\psi_2^-}{C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x}\right]\mathrm{d}x\\
&= \frac{\dot{C}(t)\psi_1^+}{C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x}\int\frac{\psi^+\psi_2^-\left(C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x\right)-\psi_1^+\psi_2^-\left(\int\psi^+\psi_2^-\,\mathrm{d}x\right)}{\left(C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x\right)^2}\mathrm{d}x\\
&= \frac{\dot{C}(t)\psi_1^+}{C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x}\int\mathrm{d}\left(\frac{\int\psi^+\psi_2^-\,\mathrm{d}x}{C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x}\right)\\
&= \frac{\dot{C}(t)\psi_1^+}{C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x}\,\frac{\int\psi^+\psi_2^-\,\mathrm{d}x}{C(t)+\int\psi_1^+\psi_2^-\,\mathrm{d}x}\\
&= \text{the rhs of (3.24)}
\end{aligned} \tag{3.25}$$

This completes the proof. □

Remark

(1) For system (3.7), the DTs described in this section can also be constructed, we omit the results here.
(2) (3.21*b*)–(3.21*e*) offer us a non-auto-Bäcklund transformation between two KPESCSs with degrees m and $m+1$, respectively. This DT can be used to construct solutions of the KPESCS (3.5).

Example 1. Single-soliton solution of KPI equation with self-consistent sources (KPIESCS). If we set $\alpha = i$ in equation (3.5), we get the KPIESCS [15]

$$\left[u_t + 6uu_x + u_{xxx} + 8\sum_{j=1}^{m}(\Phi_j\Psi_j)_x\right]_x - 3u_{yy} = 0 \tag{3.26a}$$

$$\mathrm{i}\Phi_{j,y} = \Phi_{j,xx} + u\Phi_j \tag{3.26b}$$

$$\mathrm{i}\Psi_{j,y} = -\Psi_{j,xx} - u\Psi_j \qquad j = 1,\ldots,m. \tag{3.26c}$$

We take $u = 0$ as the initial solution of (3.26) with $m = 0$ and let

$$\psi_1^+ = \mathrm{e}^{k_1x+\mathrm{i}k_1^2y-4k_1^3t} \qquad \psi_2^- = \mathrm{e}^{k_2x-\mathrm{i}k_2^2y-4k_2^3t} \qquad k_1 = \mu+\mathrm{i}\nu \qquad k_2 = \mu-\mathrm{i}\nu \qquad \mu,\nu\in\mathbb{R} \tag{3.27a}$$

$$C(t) = \frac{1}{k_1+k_2}\mathrm{e}^{2\beta(t)} = \frac{1}{2\mu}\mathrm{e}^{2\beta(t)} \tag{3.27b}$$

where $\beta(t)$ is an arbitrary function in t. Then $\psi_1^+ = \mathrm{e}^{\omega+\theta}$, $\psi_2^- = \mathrm{e}^{-\omega+\theta}$ where

$\omega = -\mathrm{i}\nu x + \mathrm{i}(\mu^2-\nu^2)y + 4\mathrm{i}(3\mu^2\nu-\nu^3)t \qquad \theta = \mu x + 2\mu\nu y - 4(\mu^3 - 3\mu\nu^2)t.$

The single-soliton solution of the KPIESCS (3.26) with $m = 1$ is given by

$$u[-1,+1] = 2\partial^2 \ln\left(C(t) + \int \psi_1^+ \psi_2^- \,\mathrm{d}x\right)$$
$$= 2\partial^2 \ln\left(\frac{1}{2\mu}\,\mathrm{e}^{2\beta(t)} + \frac{1}{2\mu}\,\mathrm{e}^{2\theta(t)}\right) = 2\mu^2 \operatorname{sech}^2(\theta - \beta(t)) \qquad (3.28a)$$

$$\Phi_1[-1,+1] = \frac{1}{2}\frac{\sqrt{\dot{C}(t)}\psi_2^-}{C(t) + \frac{1}{2\mu}\,\mathrm{e}^{2\theta(t)}} = a\,\mathrm{e}^{-\omega}\operatorname{sech}(\theta - \beta(t)) \qquad a = \frac{1}{2}\sqrt{\mu\dot{\beta}(t)} \qquad (3.28b)$$

$$\Psi_1[-1,+1] = \overline{\Phi_1[-1,+1]} = a\,\mathrm{e}^{\omega}\operatorname{sech}(\theta - \beta(t)). \qquad (3.28c)$$

We can compare the results above with those in [15].

Case 1: If we set

$$\theta - \beta(t) = \mu x + 2\mu\nu y - 4(\mu^3 - 3\mu\nu^2)t - \beta(t) = \mu(x + 2\nu y - \tau t)$$

then

$$\beta(t) = [\tau - 4(\mu^2 - 3\nu^2)]\mu t \qquad \dot{\beta}(t) = \mu[\tau - 4(\mu^2 - 3\nu^2)]$$

then we have the following identity:

$$a^2 = \frac{1}{4}\mu\dot{\beta}(t) = \frac{\mu^2}{4}[\tau - 4(\mu^2 - 3\nu^2)] \qquad (3.29)$$

which gives the relation of the parameters a, τ, μ, ν appearing in solution (3.28). Solution (3.28) together with (3.29) is also obtained in [15].

Case 2. More generally, if we set

$$\theta - \beta(t) = \mu(x + 2\nu y - f(t))$$

then

$$f(t) = 4(\mu^2 - 3\nu^2)t + \frac{\beta(t)}{\mu}$$

then the relation identity of the parameters is

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = 4(\mu^2 - 3\nu^2) + \frac{\dot{\beta}(t)}{\mu} = 4(\mu^2 - 3\nu^2) + 4\mu^{-2}a^2 \qquad (3.30)$$

which is also given in [15].

From cases 1 and 2, we see that the solutions obtained through the generalized binary DT are consistent with those in [15], and the method used here is more natural and simple.

Example 2. Lump solution (rational solution) of the KPIESCS.

We take $u = 0$, $\Phi_1 = \Psi_1 = 1$ as the initial solution of (3.26) with $m = 1$ and let

$$\psi_1^+ = \left(x + 2\mathrm{i}ky - 12k^2 t - \frac{4}{k^2}t\right)\mathrm{e}^{kx + \mathrm{i}k^2 y - 4k^3 t + \frac{4}{k}t} \qquad \psi_2^- = \overline{\psi_1^+} \quad k \in \mathbb{C}.$$

Set $k = \mu + \mathrm{i}\nu$, $\mu, \nu \in \mathbb{R}$, then

$$\psi_1^+ = \mathrm{e}^{\theta_1 + \mathrm{i}\theta_2}(\alpha_1 + \alpha_2\mathrm{i}) \qquad \psi_2^- = \mathrm{e}^{\theta_1 - \mathrm{i}\theta_2}(\alpha_1 - \alpha_2\mathrm{i})$$

where

$$\theta_1 = \mu x - 2\mu\nu y - 4\mu(\mu^2 - 3\nu^2)t + \frac{4\mu}{\mu^2+\nu^2}t$$

$$\theta_2 = \nu x + (\mu^2 - \nu^2)y - 4\nu(3\mu^2 - \nu^2)t - \frac{4\nu}{\mu^2+\nu^2}t$$

$$\alpha_1 = x - 2\nu y - 12(\mu^2-\nu^2)t - 4\frac{\mu^2-\nu^2}{(\mu^2+\nu^2)^2}t \qquad \alpha_2 = 2\mu y - 24\mu\nu t + \frac{8\mu\nu}{(\mu^2+\nu^2)^2}t.$$

By DT (3.21) with $C(t) \equiv 0$, we get the lump solution of the KPIESCS (3.26) with $m = 1$ as follows:

$$u[-1,+1] = 2\partial^2 \ln \int \psi_1^+ \psi_2^- \, \mathrm{d}x = 2\partial^2 \ln \left[\alpha_2^2 + \left(\alpha_1 - \frac{1}{2\mu}\right)^2 + \frac{1}{4\mu^2}\right] \tag{3.31a}$$

$$\begin{aligned}\Phi_1[-1,+1] &= -\Phi_1 - \frac{\psi_2^- \int \psi_1^+ \Phi_1 \, \mathrm{d}x}{\int \psi_1^+ \psi_2^- \, \mathrm{d}x} \\ &= -1 - 2\mu\left\{\left(\alpha_1^2 + \alpha_2^2\right)\mu(\mu^2+\nu^2) - \alpha_1(\mu^2-\nu^2) + 2\mu\nu\alpha_2\right. \\ &\quad \left. + \left[-\left(\alpha_1^2+\alpha_2^2\right)(\nu\mu^2+\nu^3) + (\mu^2-\nu^2)\alpha_2 + 2\mu\nu\alpha_1\right]\mathrm{i}\right\} \\ &\quad \times \left\{(\mu^2+\nu^2)^2\left[\alpha_2^2 + \left(\alpha_1 - \frac{1}{2\mu}\right)^2 + \frac{1}{4\mu^2}\right]\right\}^{-1}\end{aligned} \tag{3.31b}$$

$$\begin{aligned}\Psi_1[-1,+1] &= \overline{\Phi_1[-1,+1]}\mu \\ &= 1 - 2\mu\left\{\left(\alpha_1^2 + \alpha_2^2\right)\mu(\mu^2+\nu^2) - \alpha_1(\mu^2-\nu^2) + 2\mu\nu\alpha_2\right. \\ &\quad \left. - \left[-\left(\alpha_1^2+\alpha_2^2\right)(\nu\mu^2+\nu^3) + (\mu^2-\nu^2)\alpha_2 + 2\mu\nu\alpha_1\right]\mathrm{i}\right\} \\ &\quad \times \left\{(\mu^2+\nu^2)^2\left[\alpha_2^2 + \left(\alpha_1 - \frac{1}{2\mu}\right)^2 + \frac{1}{4\mu^2}\right]\right\}^{-1}.\end{aligned} \tag{3.31c}$$

Example 3. Single-soliton solution of the KPII equation with self-consistent sources (KPIIESCS).

If we set $\alpha = 1$ in equation (3.5), we get the KPIIESCS [16]

$$\left[u_t + 6uu_x + u_{xxx} + 8\sum_{j=1}^{m}(\Phi_j\Psi_j)_x\right]_x + 3u_{yy} = 0 \tag{3.32a}$$

$$\Phi_{j,y} = \Phi_{j,xx} + u\Phi_j \tag{3.32b}$$

$$\Psi_{j,y} = -\Psi_{j,xx} - u\Psi_j \qquad j = 1,\ldots,m. \tag{3.32c}$$

We take $u = 0$ as the initial solution of (3.32) with $m = 0$ and let

$$\psi_1^+ = \mathrm{e}^{k_1x - k_1^2y - 4k_1^3t} = \mathrm{e}^{\xi_1} \qquad \psi_2^- = \mathrm{e}^{k_2x + k_2^2y - 4k_2^3t} = \mathrm{e}^{\xi_2} \qquad k_1, k_2 \in \mathbb{R} \tag{3.33a}$$

$$C(t) = \frac{1}{k_1+k_2}\mathrm{e}^{2\beta(t)} \tag{3.33b}$$

where $\beta(t)$ is an arbitrary function in t. The single-soliton solution of the KPIIESCS (3.32) with $m = 1$ is given by

$$
\begin{aligned}
u[-1,+1] &= 2\partial^2 \ln\left(C(t) + \int \psi_1^+ \psi_2^- \,\mathrm{d}x\right) = 2\partial^2 \ln(1 + \mathrm{e}^{\xi_1+\xi_2-2\beta(t)}) \\
\Phi_1[-1,+1] &= \frac{1}{2}\frac{\sqrt{\dot{C}(t)}\psi_2^-}{C(t) + \frac{1}{k_1+k_2}\mathrm{e}^{\xi_1+\xi_2}} = \frac{\sqrt{2(k_1+k_2)\dot{\beta}}\,\mathrm{e}^{\xi_2-\beta(t)}}{2(1+\mathrm{e}^{\xi_1+\xi_2-2\beta(t)})} \\
\Psi_1[-1,+1] &= \frac{1}{2}\frac{\sqrt{\dot{C}(t)}\psi_1^+}{C(t) + \frac{1}{k_1+k_2}\mathrm{e}^{\xi_1+\xi_2}} = \frac{\sqrt{2(k_1+k_2)\dot{\beta}}\,\mathrm{e}^{\xi_1-\beta(t)}}{2(1+\mathrm{e}^{\xi_1+\xi_2-2\beta(t)})}.
\end{aligned}
\tag{3.34}
$$

4. The *N*-times repeated generalized binary Darboux transformation for the KPESCS

Assuming $f_1, \ldots, f_n$ are n arbitrary solutions of (3.6), $g_1, \ldots, g_n$ are n arbitrary solutions of (3.7), $C_1(t), \ldots, C_n(t)$ are n arbitrary functions in t, we define the following Wronskians:

$$
\begin{aligned}
&W_1(f_1, \ldots, f_n; g_1, \ldots, g_n; C_1(t), \ldots, C_n(t)) = \det(U_{n\times n}) \\
&W_2(f_1, \ldots, f_n; g_1, \ldots, g_{n-1}; C_1(t), \ldots, C_{n-1}(t)) = \det(V_{n\times n}) \\
&W_3(f_1, \ldots, f_{n-1}; g_1, \ldots, g_n; C_1(t), \ldots, C_{n-1}(t)) = \det(X_{n\times n})
\end{aligned}
\tag{4.1}
$$

where

$$
U_{i,j} = \delta_{i,j} C_i(t) + \int g_i f_j \,\mathrm{d}x \qquad i, j = 1, \ldots, n \tag{4.2a}
$$

$$
\begin{aligned}
&V_{i,j} = \delta_{i,j} C_i(t) + \int g_i f_j \,\mathrm{d}x \qquad i = 1, \ldots, n-1 \quad j = 1, \ldots, n \\
&V_{n,j} = f_j \qquad j = 1, \ldots, n
\end{aligned}
\tag{4.2b}
$$

$$
\begin{aligned}
&X_{i,j} = \delta_{i,j} C_i(t) + \int g_j f_i \,\mathrm{d}x \qquad i = 1, \ldots, n-1 \quad j = 1, \ldots, n \\
&X_{n,j} = g_j \qquad j = 1, \ldots, n.
\end{aligned}
\tag{4.2c}
$$

Lemma 4.1. *Assume $\psi^+, \psi_1^+, \ldots, \psi_N^+$ are solutions of (3.6) and $\psi^-, \psi_{N+1}^-, \ldots, \psi_{N+N}^-$ are solutions of (3.7). For notational simplicity, we denote $\psi_{l+k}^+[l] = \psi_{l+k}^+[-l,+l]$, $\psi_{N+l+k}^-[l] = \psi_{N+l+k}^-[-l,+l]$, $0 \leqslant l \leqslant N$, $1 \leqslant k \leqslant N-l$. Then*

$$
\begin{aligned}
&W_1\big(\psi_m^+[m-1], \ldots, \psi_{m+k}^+[m-1]; \psi_{N+m}^-[m-1], \ldots, \psi_{N+m+k}^-[m-1]; C_m(t), \ldots, C_{m+k}(t)\big) \\
&= \frac{W_1\big(\psi_{m-1}^+[m-2], \ldots, \psi_{m+k}^+[m-2]; \psi_{N+m-1}^-[m-2], \ldots, \psi_{N+m+k}^-[m-2]; C_{m-1}(t), \ldots, C_{m+k}(t)\big)}{C_{m-1}(t) + \int \psi_{m-1}^+[m-2]\psi_{N+m-1}^-[m-2]\,\mathrm{d}x}
\end{aligned}
\tag{4.3a}
$$

$$
\begin{aligned}
&W_2\big(\psi_m^+[m-1], \ldots, \psi_{m+k}^+[m-1], \psi^+[m-1]; \psi_{N+m}^-[m-1], \ldots, \psi_{N+m+k}^-[m-1]; \\
&C_m(t), \ldots, C_{m+k}(t)\big) \\
&= \frac{W_2\big(\psi_{m-1}^+[m-2], \ldots, \psi_{m+k}^+[m-2], \psi^+[m-2]; \psi_{N+m-1}^-[m-2], \ldots, \psi_{N+m+k}^-[m-2]; C_{m-1}(t), \ldots, C_{m+k}(t)\big)}{C_{m-1}(t) + \int \psi_{m-1}^+[m-2]\psi_{N+m-1}^-[m-2]\,\mathrm{d}x}
\end{aligned}
\tag{4.3b}
$$

$$
\begin{aligned}
&W_3\big(\psi_m^+[m-1], \ldots, \psi_{m+k}^+[m-1]; \psi_{N+m}^-[m-1], \ldots, \psi_{N+m+k}^-[m-1], \psi^-[m-1]; \\
&C_m(t), \ldots, C_{m+k}(t)\big) \\
&= \frac{W_3\big(\psi_{m-1}^+[m-2], \ldots, \psi_{m+k}^+[m-2]; \psi_{N+m-1}^-[m-2], \ldots, \psi_{N+m+k}^-[m-2], \psi^-[m-2]; C_{m-1}(t), \ldots, C_{m+k}(t)\big)}{C_{m-1}(t) + \int \psi_{m-1}^+[m-2]\psi_{N+m-1}^-[m-2]\,\mathrm{d}x}.
\end{aligned}
\tag{4.3c}
$$

Proof. According to (3.21), we have

$$\psi^+_{m-1+j}[m-1] = \psi^+_{m-1+j}[m-2] - \frac{\psi^+_{m-1}[m-2]\int\psi^-_{N+m-1}[m-2]\psi^+_{m-1+j}[m-2]\,\mathrm{d}x}{C_{m-1}(t)+\int\psi^+_{m-1}[m-2]\psi^-_{N+m-1}[m-2]\,\mathrm{d}x} \tag{4.4a}$$

$$\psi^-_{N+m-1+i}[m-1] = \psi^-_{N+m-1+i}[m-2] - \frac{\psi^-_{N+m-1}[m-2]\int\psi^+_{m-1}[m-2]\psi^-_{N+m-1+i}[m-2]\,\mathrm{d}x}{C_{m-1}(t)+\int\psi^+_{m-1}[m-2]\psi^-_{N+m-1}[m-2]\,\mathrm{d}x}. \tag{4.4b}$$

By definition (4.2a) and equations (4.4), we have

$$\begin{aligned} &W_1\big(\psi^+_m[m-1],\ldots,\psi^+_{m+k}[m-1];\psi^-_{N+m}[m-1],\ldots,\psi^-_{N+m+k}[m-1];C_m(t),\ldots,C_{m+k}(t)\big)\\ &\qquad = \det(U_{i,j}) \qquad 1\leqslant i,j\leqslant k+1 \end{aligned} \tag{4.5}$$

$$\begin{aligned} U_{i,j} &= \delta_{i,j}C_{m-1+i}(t) + \int\psi^+_{m-1+j}[m-1]\psi^-_{N+m-1+i}[m-1]\,\mathrm{d}x\\ &= \delta_{i,j}C_{m-1+i}(t) + \int\psi^+_{m-1+j}[m-2]\psi^-_{N+m-1+i}[m-2]\,\mathrm{d}x\\ &\quad - \frac{1}{C_{m-1}(t)+\int\psi^+_{m-1}[m-2]\psi^-_{N+m-1}[m-2]\,\mathrm{d}x}\\ &\quad\times\left(\int\psi^+_{m-1+j}[m-2]\psi^-_{N+m-1}[m-2]\,\mathrm{d}x\right)\left(\int\psi^+_{m-1}[m-2]\psi^-_{N+m-1+i}[m-2]\,\mathrm{d}x\right)\\ &= \delta_{i,j}C_{m-1+i}(t) + a_{i,j} - b a_{i,0}a_{0,j} \end{aligned} \tag{4.6}$$

where

$$a_{i,j} = \int\psi^+_{m-1+j}[m-2]\psi^-_{N+m-1+i}[m-2]\,\mathrm{d}x$$

$$b = \frac{1}{C_{m-1}(t)+\int\psi^+_{m-1}[m-2]\psi^-_{N+m-1}[m-2]\,\mathrm{d}x}.$$

Then

$$W_1\big(\psi^+_m[m-1],\ldots,\psi^+_{m+k}[m-1];\psi^-_{N+m}[m-1],\ldots,\psi^-_{N+m+k}[m-1];C_m(t),\ldots,C_{m+k}(t)\big) =$$

$$\begin{vmatrix} C_m(t)+a_{1,1}-ba_{1,0}a_{0,1} & a_{1,2}-ba_{1,0}a_{0,2} & \cdots & a_{1,k+1}-ba_{1,0}a_{0,k+1}\\ a_{2,1}-ba_{2,0}a_{0,1} & C_{m+1}(t)+a_{2,2}-ba_{2,0}a_{0,2} & \cdots & a_{2,k+1}-ba_{2,0}a_{0,k+1}\\ \vdots & \vdots & \ddots & \vdots\\ a_{k+1,1}-ba_{k+1,0}a_{0,1} & a_{k+1,2}-ba_{k+1,0}a_{0,2} & \cdots & C_{m+k}(t)+a_{k+1,k+1}-ba_{k+1,0}a_{0,k+1} \end{vmatrix}$$

$$= \begin{vmatrix} C_m(t)+a_{1,1} & a_{1,2} & \cdots & a_{1,k+1}\\ a_{2,1} & C_{m+1}(t)+a_{2,2} & \cdots & a_{2,k+1}\\ \vdots & \vdots & \ddots & \vdots\\ a_{k+1,1} & a_{k+1,2} & \cdots & C_{m+k}(t)+a_{k+1,k+1} \end{vmatrix}$$

$$- ba_{0,1}\begin{vmatrix} a_{1,0} & a_{1,2} & \cdots & a_{1,k+1}\\ a_{2,0} & C_{m+1}(t)+a_{2,2} & \cdots & a_{2,k+1}\\ \vdots & \vdots & \ddots & \vdots\\ a_{k+1,0} & a_{k+1,2} & \cdots & C_{m+k}(t)+a_{k+1,k+1} \end{vmatrix}$$

$$
- ba_{0,2}\begin{vmatrix} C_m(t)+a_{1,1} & a_{1,0} & \cdots & a_{1,k+1} \\ a_{2,1} & a_{2,0} & \cdots & a_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,1} & a_{k+1,0} & \cdots & C_{m+k}(t)+a_{k+1,k+1} \end{vmatrix}
$$

$$
- \cdots - ba_{0,k+1}\begin{vmatrix} C_m(t)+a_{1,1} & a_{1,2} & \cdots & a_{1,k} & a_{1,0} \\ a_{2,1} & C_{m+1}(t)+a_{2,2} & \cdots & a_{2,k} & a_{2,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k} & a_{k+1,0} \end{vmatrix}
$$

$$
= b\begin{vmatrix} C_{m-1}(t)+a_{0,0} & a_{0,1} & a_{0,2} & \cdots & a_{0,k+1} \\ a_{1,0} & C_m(t)+a_{1,1} & a_{1,2} & \cdots & a_{1,k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k+1,0} & a_{k+1,1} & a_{k+1,2} & \cdots & C_{m+k}(t)+a_{k+1,k+1} \end{vmatrix} =
$$

$$
\frac{W_1\left(\psi^+_{m-1}[m-2],\ldots,\psi^+_{m+k}[m-2];\psi^-_{N+m-1}[m-2],\ldots,\psi^-_{N+m+k}[m-2];C_{m-1}(t),\ldots,C_{m+k}(t)\right)}{C_{m-1}(t)+\int\psi^+_{m-1}[m-2]\psi^-_{N+m-1}[m-2]\,\mathrm{d}x}.
$$

So formula (4.3a) holds. Formulae (4.3b) and (4.3c) can be proved similarly. This completes the proof. □

Theorem 4.1. *Assume that* $u, \Psi_1, \ldots, \Psi_m, \Phi_1, \ldots, \Phi_m$ *is a solution of the KPESCS (3.5),* $\psi^+_1, \ldots, \psi^+_N$ *and* $\psi^-_{N+1}, \ldots, \psi^-_{N+N}$ *are solutions of (3.6) and (3.7) respectively,* $C_1(t), \ldots, C_N(t)$ *are N arbitrary functions in t. Then the N-times repeated generalized binary Darboux transformation for (3.6) is given by*

$$
\psi^+[-N,+N] = \frac{W_2\left(\psi^+_1,\ldots,\psi^+_N,\psi^+;\psi^-_{N+1},\ldots,\psi^-_{N+N};C_1(t),\ldots,C_N(t)\right)}{W_1\left(\psi^+_1,\ldots,\psi^+_N;\psi^-_{N+1},\ldots,\psi^-_{N+N};C_1(t),\ldots,C_N(t)\right)} \tag{4.7a}
$$

$$
u[-N,+N] = u + 2\partial^2\ln W_1\left(\psi^+_1,\ldots,\psi^+_N;\psi^-_{N+1},\ldots,\psi^-_{N+N};C_1(t),\ldots,C_N(t)\right) \tag{4.7b}
$$

$$
\Psi_l[-N,+N] = -\frac{W_2\left(\psi^+_1,\ldots,\psi^+_N,\Psi_l;\psi^-_{N+1},\ldots,\psi^-_{N+N};C_1(t),\ldots,C_N(t)\right)}{W_1\left(\psi^+_1,\ldots,\psi^+_N;\psi^-_{N+1},\ldots,\psi^-_{N+N};C_1(t),\ldots,C_N(t)\right)} \tag{4.7c}
$$

$$
\Phi_l[-N,+N] = -\frac{W_3\left(\psi^+_1,\ldots,\psi^+_N;\psi^-_{N+1},\ldots,\psi^-_{N+N},\Phi_l;C_1(t),\ldots,C_N(t)\right)}{W_1\left(\psi^+_1,\ldots,\psi^+_N;\psi^-_{N+1},\ldots,\psi^-_{N+N};C_1(t),\ldots,C_N(t)\right)}
$$

$$
l = 1,\ldots,m \tag{4.7d}
$$

$$
\Psi_{m+j}[-N,+N] = \frac{1}{2}\sqrt{\dot{C}_j(t)}
$$

$$
\times\frac{W_2\left(\psi^+_1,\ldots,\psi^+_{j-1},\psi^+_{j+1},\ldots,\psi^+_N,\psi^+_j;\psi^-_{N+1},\ldots,\psi^-_{N+j-1},\psi^-_{N+j+1},\ldots,\psi^-_{N+N};C_1(t),\ldots,C_{j-1}(t),C_{j+1}(t),\ldots,C_N(t)\right)}{W_1\left(\psi^+_1,\ldots,\psi^+_N;\psi^-_{N+1},\ldots,\psi^-_{N+N};C_1(t),\ldots,C_N(t)\right)} \tag{4.7e}
$$

$$
\Phi_{m+j}[-N,+N] = \frac{1}{2}\sqrt{\dot{C}_j(t)}
$$

$$
\times\frac{W_3\left(\psi^+_1,\ldots,\psi^+_{j-1},\psi^+_{j+1},\ldots,\psi^+_N;\psi^-_{N+1},\ldots,\psi^-_{N+j-1},\psi^-_{N+j+1},\ldots,\psi^-_{N+N},\psi^-_{N+j};C_1(t),\ldots,C_{j-1}(t),C_{j+1}(t),\ldots,C_N(t)\right)}{W_1\left(\psi^+_1,\ldots,\psi^+_N;\psi^-_{N+1},\ldots,\psi^-_{N+N};C_1(t),\ldots,C_N(t)\right)}
$$

$$
j = 1,\ldots,N. \tag{4.7f}
$$

Namely,

$$\alpha\Psi_j^+[-N,+N]_y = -\Psi_j^+[-N,+N]_{xx} - u[-N,+N]\Psi_j^+[-N,+N] \tag{4.8a}$$

$$\alpha\Phi_j^-[-N,+N]_y = \Phi_j^-[-N,+N]_{xx} + u[-N,+N]\Phi_j^-[-N,+N],\qquad j=1,\ldots,m+N \tag{4.8b}$$

$$\alpha\psi^+[-N,+N]_y = -\psi^+[-N,+N]_{xx} - u[-N,+N]\psi^+[-N,+N] \tag{4.8c}$$

$$\begin{aligned}\psi^+[-N,+N]_t &= A^+(u[-N,+N])\psi^+[-N,+N]\\ &\quad + T^+_{m+N}(\Psi[-N,+N],\Phi[-N,+N])\psi^+[-N,+N].\end{aligned} \tag{4.8d}$$

So $u[-N,+N]$, $\Psi_j[-N,+N]$, $\Phi_j[-N,+N]$, $j=1,\ldots,m+N$ *satisfy the KPESCS (3.5) with degree* $(m+N)$.

Proof. For notational simplicity, we denote $\psi^+[-l,+l]$, $u[-l,+l]$, $\Psi_i[-l,+l]$, $\Phi_i[-l,+l]$ by $\psi^+[l]$, $u[l]$, $\Psi_i[l]$, $\Phi_i[l]$ respectively, $0\leqslant l\leqslant N$, $1\leqslant i\leqslant m+N$. By (3.21) and (4.3),

$$\begin{aligned}\psi^+[N] &= \psi^+[N-1] - \frac{\psi_N^+[N-1]\int\psi_{N+N}^-[N-1]\psi^+[N-1]\,\mathrm{d}x}{C_N(t)+\int\psi_{N+N}^-[N-1]\psi_N^+[N-1]\,\mathrm{d}x}\\
&= \frac{W_2\big(\psi_N^+[N-1],\psi^+[N-1];\psi_{N+N}^-[N-1];C_N(t)\big)}{W_1\big(\psi_N^+[N-1];\psi_{N+N}^-[N-1];C_N(t)\big)}\\
&= \frac{W_2\big(\psi_{N-1}^+[N-2],\psi_N^+[N-2],\psi^+[N-2];\psi_{N+N-1}^-[N-2],\psi_{N+N}^-[N-2];C_{N-1}(t),C_N(t)\big)}{C_{N-1}(t)+\int\psi_{N+N-1}^-[N-2]\psi_{N-1}^+[N-2]\,\mathrm{d}x}\\
&\quad\times\frac{C_{N-1}(t)+\int\psi_{N+N-1}^-[N-2]\psi_{N-1}^+[N-2]\,\mathrm{d}x}{W_1\big(\psi_{N-1}^+[N-2],\psi_N^+[N-2];\psi_{N+N-1}^-[N-2],\psi_{N+N}^-[N-2];C_{N-1}(t),C_N(t)\big)}\\
&= \cdots\\
&= \frac{W_2\big(\psi_1^+,\ldots,\psi_N^+,\psi^+;\psi_{N+1}^-,\ldots,\psi_{N+N}^-;C_1(t),\ldots,C_N(t)\big)}{W_1\big(\psi_1^+,\ldots,\psi_N^+;\psi_{N+1}^-,\ldots,\psi_{N+N}^-;C_1(t),\ldots,C_N(t)\big)},\end{aligned}$$

so (4.7a) holds, Similarly we can prove that (4.7c) and (4.7d) hold,

$$\begin{aligned}u[N] &= u[N-1] + 2\partial^2\ln\left(C_N(t)+\int\psi_N^+[N-1]\psi_{N+N}^-[N-1]\,\mathrm{d}x\right)\\
&= u[N-1] + 2\partial^2\ln W_1\big(\psi_N^+[N-1];\psi_{N+N}^-[N-1];C_N(t)\big)\\
&= u[N-2] + 2\partial^2\ln\left(C_{N-1}(t)+\int\psi_{N-1}^+[N-2]\psi_{N+N-1}^-[N-2]\,\mathrm{d}x\right) + 2\partial^2\ln\\
&\qquad\times\left[\frac{W_1\big(\psi_{N-1}^+[N-2],\psi_N^+[N-2];\psi_{N+N-1}^-[N-2],\psi_{N+N}^-[N-2];C_{N-1}(t),C_N(t)\big)}{C_{N-1}(t)+\int\psi_{N-1}^+[N-2]\psi_{N+N-1}^-[N-2]\,\mathrm{d}x}\right]\\
&= u[N-2] + 2\partial^2\ln W_1\big(\psi_{N-1}^+[N-2],\psi_N^+[N-2];\psi_{N+N-1}^-[N-2],\psi_{N+N}^-[N-2];\\
&\qquad C_{N-1}(t),C_N(t)\big)\\
&= \cdots\\
&= u + 2\partial^2\ln W_1\big(\psi_1^+,\ldots,\psi_N^+;\psi_{N+1}^-,\ldots,\psi_{N+N}^-;C_1(t),\ldots,C_N(t)\big)\end{aligned}$$

so (4.7b) holds.

From (3.21a) and (3.21e), we have

$$\psi_j^+[j] = \psi_j^+[j-1] - \frac{\psi_j^+[j-1]\int \psi_j^+[j-1]\psi_{N+j}^-[j-1]\,\mathrm{d}x}{C_j(t) + \int \psi_j^+[j-1]\psi_{N+j}^-[j-1]\,\mathrm{d}x}$$
$$= \frac{C_j(t)\psi_j^+[j-1]}{C_j(t) + \int \psi_j^+[j-1]\psi_{N+j}^-[j-1]\,\mathrm{d}x}$$

$$\Psi_{m+j}[j] = \frac{1}{2}\frac{\sqrt{\dot{C}_j(t)}\psi_j^+[j-1]}{C_j(t) + \int \psi_j^+[j-1]\psi_{N+j}^-[j-1]\,\mathrm{d}x}$$

so

$$\Psi_{m+j}[j] = \frac{1}{2}\frac{\sqrt{\dot{C}_j(t)}}{C_j(t)}\psi_j^+[j].$$

Analogously as above, we have

$$\Psi_{m+j}[N]$$
$$= \frac{W_2\big(\psi_N^+[N-1], \Psi_{m+j}[N-1]; \psi_{N+N}^-[N-1]; C_N(t)\big)}{W_1\big(\psi_N^+[N-1]; \psi_{N+N}^-[N-1]; C_N(t)\big)}$$
$$= \frac{W_2\big(\psi_{j+1}^+[j], \ldots, \psi_N^+[j], \Psi_{m+j}[j]; \psi_{N+j+1}^-[j], \ldots, \psi_{N+N}^-[j]; C_{j+1}(t), \ldots, C_N(t)\big)}{W_1\big(\psi_{j+1}^+[j], \ldots, \psi_N^+[j]; \psi_{N+j+1}^-[j], \ldots, \psi_{N+N}^-[j]; C_{j+1}(t), \ldots, C_N(t)\big)}$$
$$= \frac{\sqrt{\dot{C}_j(t)}}{2C_j(t)}\frac{W_2\big(\psi_{j+1}^+[j], \ldots, \psi_N^+[j], \psi_j^+[j]; \psi_{N+j+1}^-[j], \ldots, \psi_{N+N}^-[j]; C_{j+1}(t), \ldots, C_N(t)\big)}{W_1\big(\psi_{j+1}^+[j], \ldots, \psi_N^+[j]; \psi_{N+j+1}^-[j], \ldots, \psi_{N+N}^-[j]; C_{j+1}(t), \ldots, C_N(t)\big)}$$
$$= \frac{\sqrt{\dot{C}_j(t)}}{2C_j(t)}\frac{W_2\big(\psi_1^+, \ldots, \psi_N^+, \psi_j^+; \psi_{N+1}^-, \ldots, \psi_{N+N}^-; C_1(t), \ldots, C_N(t)\big)}{W_1\big(\psi_1^+, \ldots, \psi_N^+; \psi_{N+1}^-, \ldots, \psi_{N+N}^-; C_1(t), \ldots, C_N(t)\big)}$$
$$= \frac{1}{2}\sqrt{\dot{C}_j(t)}$$
$$\times \frac{W_2\big(\psi_1^+,\ldots,\psi_{j-1}^+,\psi_{j+1}^+,\ldots,\psi_N^+,\psi_j^+;\psi_{N+1}^-,\ldots,\psi_{N+j-1}^-,\psi_{N+j+1}^-,\ldots,\psi_{N+N}^-;C_1(t),\ldots,C_{j-1}(t),C_{j+1}(t),\ldots,C_N(t)\big)}{W_1\big(\psi_1^+,\ldots,\psi_N^+;\psi_{N+1}^-,\ldots,\psi_{N+N}^-;C_1(t),\ldots,C_N(t)\big)}$$

so (4.7e) holds. In a similar way we can prove (4.7f) holds. This completes the proof. □

Remark

(1) If $C_1(t), \ldots, C_N(t)$ are replaced by N arbitrary constants $C_1, \ldots, C_N$, then (4.7) gives the N-times repeated Darboux transformation for (3.18).
(2) The N-times repeated generalized binary Darboux transformation (4.7) provides a non-auto Bäcklund transformation between two KPESCSs with degrees m and $(m + N)$, respectively. We now use this to construct some interesting solutions for the KPESCS.

Example 4. (Degenerate case) When some $C_j(t)$ in (4.7) are chosen to be arbitrary numerical constants C_j, we can get some interesting solutions for the KPESCS (3.5) in the degenerate case ('degenerate' means the number of solitons is larger than the degree of the equation). In the following, we will take the KPIIESCS (3.32) for example and choose $u = 0$ as the initial solution of it with $m = 0$.

Example 4.1. Two-soliton solution of the KPIIESCS.

Take

$$\psi_i^+ = \mathrm{e}^{\eta_i} \qquad \psi_{2+i}^- = \mathrm{e}^{\theta_i} \qquad i = 1, 2$$
$$\eta_i = l_i x - l_i^2 y - 4l_i^3 t \qquad \theta_i = k_i x + k_i^2 y - 4k_i^3 t \qquad l_i, k_i \in \mathbb{R}$$
$$C_1(t) = \frac{1}{k_1 + l_1}\,\mathrm{e}^{2\beta(t)} \qquad C_2(t) \equiv \frac{c}{k_2 + l_2}$$

where $\beta(t)$ is an arbitrary function in t and c is an arbitrary constant. From (4.7b), (4.7e) and (4.7f), we get the 2-soliton solution of the KPIIESCS (3.32) with $m = 1$ as follows:

$$\begin{aligned}
&u(x, y, t)[-2, +2]\\
&= 2\partial^2 \ln W_1\big(\psi_1^+, \psi_2^+; \psi_3^-, \psi_4^-; C_1(t), C_2(t)\big)\\
&= 2\partial^2 \ln \begin{vmatrix} C_1(t) + \int \psi_3^- \psi_1^+ \,\mathrm{d}x & \int \psi_3^- \psi_2^+ \,\mathrm{d}x \\ \int \psi_4^- \psi_1^+ \,\mathrm{d}x & C_2(t) + \int \psi_4^- \psi_2^+ \,\mathrm{d}x \end{vmatrix}\\
&= 2\partial^2 \ln \begin{vmatrix} \frac{1}{k_1+l_1}\mathrm{e}^{2\beta(t)} + \frac{1}{k_1+l_1}\mathrm{e}^{\theta_1+\eta_1} & \frac{1}{k_1+l_2}\mathrm{e}^{\theta_1+\eta_2} \\ \frac{1}{k_2+l_1}\mathrm{e}^{\theta_2+\eta_1} & \frac{c}{k_2+l_2}\mathrm{e}^{\theta_2+\eta_2} \end{vmatrix}\\
&= 2\partial^2 \ln\left[\frac{c\mathrm{e}^{2\beta(t)}}{(k_1+l_1)(k_2+l_2)}\left(1 + \mathrm{e}^{\theta_1+\eta_1-2\beta(t)} + \mathrm{e}^{\theta_2+\eta_2+\xi_0} + \frac{(k_1-k_2)(l_1-l_2)}{(k_2+l_1)(k_1+l_2)}\,\mathrm{e}^{\theta_1+\theta_2+\eta_1+\eta_2-2\beta(t)+\xi_0}\right)\right]\\
&= 2\partial^2 \ln\left[1 + \mathrm{e}^{\theta_1+\eta_1-2\beta(t)} + \mathrm{e}^{\theta_2+\eta_2+\xi_0} + \frac{(k_1-k_2)(l_1-l_2)}{(k_2+l_1)(k_1+l_2)}\,\mathrm{e}^{\theta_1+\theta_2+\eta_1+\eta_2-2\beta(t)+\xi_0}\right]
\end{aligned}$$

where $\xi_0 = -\ln c$,

$$\begin{aligned}
&\Psi_1[-2, +2]\\
&= \frac{1}{2}\sqrt{\dot{C}_1(t)}\,\frac{W_2\big(\psi_2^+, \psi_1^+; \psi_4^-; C_2(t)\big)}{W_1\big(\psi_1^+, \psi_2^+; \psi_3^-, \psi_4^-; C_1(t), C_2(t)\big)}\\
&= \frac{1}{2}\sqrt{2(k_1+l_1)\dot{\beta}(t)}\,\mathrm{e}^{\eta_1-\beta(t)}\,\frac{1 + \frac{l_1-l_2}{k_2+l_1}\mathrm{e}^{\theta_2+\eta_2+\xi_0}}{1 + \mathrm{e}^{\theta_1+\eta_1-2\beta(t)} + \mathrm{e}^{\theta_2+\eta_2+\xi_0} + \frac{(k_1-k_2)(l_1-l_2)}{(k_2+l_1)(k_1+l_2)}\,\mathrm{e}^{\theta_1+\theta_2+\eta_1+\eta_2-2\beta(t)+\xi_0}}
\end{aligned}$$

$$\begin{aligned}
\Phi_1[-2, +2] &= \frac{1}{2}\sqrt{\dot{C}_1(t)}\,\frac{W_3\big(\psi_2^+; \psi_4^-, \psi_3^-; C_2(t)\big)}{W_1\big(\psi_1^+, \psi_2^+; \psi_3^-, \psi_4^-; C_1(t), C_2(t)\big)}\\
&= \frac{1}{2}\sqrt{2(k_1+l_1)\dot{\beta}(t)}\,\mathrm{e}^{\theta_1-\beta(t)}\,\frac{1 + \frac{k_1-k_2}{k_1+l_2}\,\mathrm{e}^{\theta_2+\eta_2+\xi_0}}{1 + \mathrm{e}^{\theta_1+\eta_1-2\beta(t)} + \mathrm{e}^{\theta_2+\eta_2+\xi_0} + \frac{(k_1-k_2)(l_1-l_2)}{(k_2+l_1)(k_1+l_2)}\,\mathrm{e}^{\theta_1+\theta_2+\eta_1+\eta_2-2\beta(t)+\xi_0}}
\end{aligned}$$

$\Psi_2[-2, +2] = \Phi_2[-1, +2] = 0.$

In a similar way, for $\forall N, m \in \mathbb{N}$, $N > m$, when $C_j(t)$, $j = 1, \ldots, m$, are taken to be arbitrary functions in t and $C_j(t)$, $j = m+1, \ldots, N$, are taken to be numerical constants, we can get the N-soliton solution of (3.32) with degree m.

Example 4.2. Mixture of the exponential and rational solutions of the KPIIESCS.

When we take

$$\psi_1^+ = \mathrm{e}^{\zeta}(x - 2qy - 12q^2 t) \qquad \psi_2^+ = \mathrm{e}^{\zeta} \qquad \zeta = qx - q^2 y - 4q^3 t \qquad q \in \mathbb{R}$$
$$\psi_3^- = \mathrm{e}^{\xi}(x + 2ky - 12k^2 t) \qquad \psi_4^- = \mathrm{e}^{\xi} \qquad \xi = kx + k^2 y - 4k^3 t \qquad k \in \mathbb{R}$$
$$C_1(t) = \frac{1}{q+k}\,\mathrm{e}^{2\beta(t)} \qquad C_2(t) \equiv \frac{c}{q+k}$$

where $\beta(t)$ is an arbitrary function in t and c is a constant, we will get another solution (mixture of the exponential and rational solutions) of the KPIIESCS with degree $m = 1$ as follows:

$$u(x, y, t)[-2, +2] = 2\partial^2 \ln \left[1 + e^{\zeta+\xi+\xi_0} + e^{\zeta+\xi-2\beta(t)} r + \frac{1}{(k+q)^2} e^{2(\zeta+\xi)-2\beta(t)+\xi_0} \right]$$

$$\Psi_1[-2, +2] = \frac{1}{2}\sqrt{2(k+q)\dot{\beta}(t)}\, e^{\zeta-\beta(t)} \frac{x - 2qy - 12q^2t + \frac{1}{k+q} e^{\zeta+\xi+\xi_0}}{1 + e^{\zeta+\xi+\xi_0} + e^{\zeta+\xi-2\beta(t)} r + \frac{1}{(k+q)^2} e^{2(\zeta+\xi)-2\beta(t)+\xi_0}}$$

$$\Phi_1[-2, +2] = \frac{1}{2}\sqrt{2(k+q)\dot{\beta}(t)}\, e^{\xi-\beta(t)} \frac{x + 2ky - 12k^2t + \frac{1}{k+q} e^{\zeta+\xi+\xi_0}}{1 + e^{\zeta+\xi+\xi_0} + e^{\zeta+\xi-2\beta(t)} r + \frac{1}{(k+q)^2} e^{2(\zeta+\xi)-2\beta(t)+\xi_0}}$$

where $\xi_0 = -\ln c$, $r = \left(x + 2ky - 12k^2t - \frac{1}{k+q}\right)\left(x - 2qy - 12q^2t - \frac{1}{k+q}\right) + \frac{1}{(k+q)^2}$.

Example 5. (Nondegenerate case) N-soliton solution.

Take the case of $\alpha = \mathrm{i}$ for example. We take $u = 0$ as the initial solution of (3.26) with $m = 0$ and let

$$\psi_j^+ = e^{k_j x + \mathrm{i}k_j^2 y - 4k_j^3 t} \qquad \psi_{N+j}^- = \overline{\psi_j^+} \qquad k_j \in \mathbb{C}$$

$$C_j(t) = \frac{1}{2\mathrm{Re}(k_j)} e^{2\beta_j(t)} \qquad j = 1, \ldots, N$$

where $\beta_j(t)$ are arbitrary functions in t. Then formulae (4.7b), (4.7e) and (4.7f) give the N-soliton solution of the KPIESCS (3.26) with degree N.

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